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LETTER TO THE EDITOR

Cluster size distribution above the percolation threshold

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Abstract. We discuss how the cluster size distribution above the percolation threshold is related to the existence of metastable states of the Potts model. We thus find that the average number of clusters per site containing n sites is given by $C(n) \sim n^{-\theta'} \exp(-\text{constant}\, n^{1-1/d})$, where d is the dimensionality of the lattice. The exponent θ' is given for a range of values of d.

In the problem of percolation, bonds (sites) on a lattice are occupied with probability p and vacant with probability $1-p$. Clusters are defined as sets of sites connected by occupied bonds (sets of contiguous occupied sites). At a critical probability p_c, an infinite cluster forms and exists for all $p > p_c$. Cluster statistics are described by the cluster size distribution function, $C(n)$, which is the average number of clusters per site containing n sites. For $p < p_c$, it is now well established that $C(n)$ satisfies the scaling equation

$$C(n) = n^{-\tau} f[(p_c - p)^{\Delta} n] \tag{1}$$

where τ and Δ are critical exponents associated with the percolation transition. The function $f(z)$ is a constant at $z = 0$, but for large z it is proportional to $z^{-(\theta-\tau)} e^{-Az}$, where A is a constant and θ is the critical exponent associated with the statistics of lattice animals (Harris and Lubensky 1980). For $p > p_c$, $C(n)$ has a structure of the form

$$C(n) = n^{-\theta'} \exp[-B(p - p_c) n^{\zeta}] \tag{2}$$

where $\zeta = 1 - 1/d$ and d is the dimensionality of the lattice. The exponent ζ is therefore not equal to unity as for $p < p_c$.

The form for $C(n)$ given by (2) was first suggested by Kunz and Souillard (1978a, b) who placed rigorous bounds on $C(n)$. Monte Carlo experiments have also verified that $\zeta = \frac{1}{2}$ in two dimensions (see Stauffer (1979) for a review). More recently, the exponential factor in equation (2) was derived by considering droplet solutions in a field theoretic formulation of the percolation problem (Harris and Lubensky 1980). The purpose of this Letter is to give a more thorough treatment of the droplet solutions, and in particular to calculate the exponent θ'.

Before discussing the droplet solutions that lead to equation (2), it is necessary to review briefly the connection between percolation and the one-state Potts model. In the Potts model, each site x on the lattice containing N sites can be in any one of s different states specified by a variable $\sigma(x)$. The Potts model Hamiltonian is then

$$H = -J \sum_{\langle x,x' \rangle} [s\delta(\sigma(x), \sigma(x')) - 1] - h \sum_{x} [s\delta(\sigma(x), 1) - 1]. \tag{3}$$

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With some manipulations, one can show that the partition function for this model is given by

$$Z_s(h) = \sum_{\mathcal{G}} P(\mathcal{G}) \prod_n [1 + (s-1)\,\mathrm{e}^{-shn}]^{NK(\mathcal{G};n)} \tag{4}$$

where $P(\mathcal{G})$ is the probability of occurrence of a particular configuration $\mathcal{G}$ of occupied and unoccupied bonds and $K(\mathcal{G}; n)$ is the number of clusters per site in configuration $\mathcal{G}$ containing n sites (Fortuin and Kasteleyn 1972, Lubensky 1979). Note that since $\Sigma_{\mathcal{G}} P(\mathcal{G}) = 1$, $Z_s(h)$ must satisfy

$$\lim_{s\to 1} Z_s(h) = 1. \tag{5}$$

The generating function for $C(n)$ is readily obtained from (4):

$$\lim_{s\to 1} \frac{1}{s-1}\frac{1}{N} \ln Z_s(h) \equiv F(h) = \sum_n C(n)\,\mathrm{e}^{-hn} \tag{6}$$

so that for large n

$$C(n) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}h\; \mathrm{e}^{hn} F(h) \tag{7}$$

where c is such that all the singularities of $F(h)$ lie to the left of the contour.

It is clear that if $C(n)$ is to satisfy equation (2), $F(h)$ must have a branch cut for $h < 0$, and so closing the contour in (7) gives

$$C(n) = \frac{1}{\pi} \int_{-\infty}^{0} \mathrm{d}h\; \mathrm{e}^{hn}\, \mathrm{Im}\, F(h) = \frac{1}{\pi} \int_0^{\infty} \mathrm{d}|h|\, \mathrm{e}^{-|h|n}\, \mathrm{Im}\, F(-|h|) \tag{8}$$

where $\ln \mathrm{Im}\, F(-|h|) \sim |h|^{-(d-1)}$. This singularity is of exactly the same form as found by Langer (1967) in his classic analysis of metastability in the Ising model in an external magnetic field h. We are thus led to a study of metastability and critical droplets in the one-state Potts model.

To proceed, we employ a now standard field theoretic representation of the Potts model

$$Z = \int D\psi_l \exp(-H[\psi_l]) \tag{9}$$

where $H[\psi_l]$ has an infinite power-series representation in terms of the $(s-1)$-dimensional field ψ_l. For our present purposes, it is sufficient to consider properties near p_c where H is well approximated by

$$H[\psi_l] = \int \mathrm{d}^d x \left(\sum_l [\tfrac{1}{2}(\nabla\psi_l)^2 + \tfrac{1}{2} r \psi_l^2] - \frac{w}{3!} \lambda_{l_1 l_2 l_3} \psi_{l_1} \psi_{l_2} \psi_{l_3} - h_l \psi_l \right) \tag{10}$$

where the repeated index summation convention is understood, $r \sim (p - p_c)$ and where

$$\lambda_{l_1 l_2 l_3} = \sum_{\alpha=1}^{s} e_{l_1}^{\alpha} e_{l_2}^{\alpha} e_{l_3}^{\alpha}, \qquad h_l = h e_l^1. \tag{11}$$

Here e_l^{α} are vectors of modulus $(s-1)^{1/2}$ satisfying (Zia and Wallace 1975)

$$\sum_{\alpha=1}^{s} e_l^{\alpha} = 0, \qquad \sum_{\alpha=1}^{s} e_l^{\alpha} e_{ll}^{\alpha} = s\delta_{ll'}, \qquad \sum_{l=1}^{s-1} e_l^{\alpha} e_l^{\alpha'} = s\delta^{\alpha\alpha'} - 1. \tag{12}$$

Fields corresponding to extrema of $H[\psi_l]$ satisfy

$$\delta H/\delta\psi_l(x) = -\nabla^2\psi_l + r\psi_l - \tfrac{1}{2}w\lambda_{ll_1l_2}\psi_{l_1}\psi_{l_2} - h_l = 0. \tag{13}$$

To locate droplet extrema, we first identify spatially uniform solutions to equation (13). Although there are many different uniform extrema, the ones of interest to us are of the form

$$\psi_l^{(1)}(x) = \psi e_l^1, \tag{14a}$$

$$\psi_l^{(\alpha)}(x) = \psi_1 e_l^1 + \psi_2 e_l^\alpha, \qquad \alpha = 2, \ldots, s-1. \tag{14b}$$

Note that there is one solution (14*a*) that is parallel to the external field h_l and $(s-1)$ that are not.

When $h = 0$, all s solutions in equation (14) are degenerate. For $r > 0$ ($p < p_c$), the only stable solution is $\psi_l = 0$. For $r = -\mu^2 < 0$ ($p > p_c$), solutions of the form (14) with

$$\begin{aligned} &\psi = 2\mu^2/w + h/\mu^2 + \mathrm{O}(h^2), \\ &\psi_1 = h/\mu^2 + \mathrm{O}(h^2), \qquad \psi_2 = 2\mu^2/w - 2h/\mu^2 + \mathrm{O}(h^2), \end{aligned} \tag{15}$$

are stable with energies

$$\begin{aligned} \varepsilon^{(1)} &= (1/\Omega)H[\psi_l^{(1)}] = -\tfrac{2}{3}(s-1)\mu^6/w^2 - (s-1)\mu^2 h/w + \mathrm{O}(h^2), \\ \varepsilon^{(\alpha)} &= (1/\Omega)H[\psi_l^{(\alpha)}] = -\tfrac{2}{3}(s-1)\mu^6/w^2 + \mu^2 h/w + \mathrm{O}(h^2), \end{aligned} \tag{16}$$

where Ω is the volume and where only terms linear in $(s-1)$ have been retained. Thus we see that for $h > 0$ $\varepsilon^{(1)} < \varepsilon^{(\alpha)}$ and the solution (14*a*) is stable, whereas for $h < 0$ $\varepsilon^{(1)} > \varepsilon^{(\alpha)}$ and the second solution (14*b*) becomes the stable one. This is the same result as was found by Harris and Lubensky (1980) using the untruncated form for H.

Now the system can condense globally into solution (14*a*), but not into solution (14*b*) since $Z_s \sim \exp(-H[\psi_l]) \to 1$ as $s \to 1$ (equation (5)) and $\varepsilon^{(1)} \to 0$ as $s \to 1$ but $\varepsilon^{(\alpha)} \not\to 0$ as $s \to 1$. Thus, for $h < 0$, the system is constrained to be in phase 1 at spatial infinity and is metastable. Droplets of any of the $(s-1)$ other phases may form, however, contributing to Im Z. Droplet solutions, $\bar{\psi}_l^{(\alpha)}(x)$, to equation (13) are thus solutions that exist for $r < 0$ and $h < 0$ and satisfy $\bar{\psi}_l^{(\alpha)}(x - x_0) \sim \psi_l^{(\alpha)}$ for $|x - x_0| < R_0$ and $\bar{\psi}_l^{(\alpha)}(x - x_0) \sim \psi_l^{(1)}$ for $|x - x_0| > R_0$, where R_0 is the radius of the droplets and x_0 marks the centre of the droplet.

The energy of a droplet is thus

$$E_D(R) = -\Delta E R^d + \sigma R^{d-1} \tag{17}$$

where $\Delta E = \varepsilon^{(\alpha)} - \varepsilon^{(1)}$ and σ is the surface tension, which is proportional to a condensation energy times a correlation length. The surface tension is finite when $h \to 0$ and is proportional to $(\mu^6/w^2)/\mu = \mu^5/w^2$. Minimising E_D with respect to R, we obtain

$$R_0 \sim \mu^3/|h|w \tag{18}$$

and thus

$$H[\bar{\psi}_l] = E_D(R_0) = D\frac{\mu^{(6-d)}}{w^2}\left(\frac{\mu^4}{|h|w}\right)^{d-1} \tag{19}$$

where D is a constant. This result was also obtained by Harris and Lubensky (1980).

Writing $\psi_l = \bar{\psi}_l + \hat{\psi}_l$ in equation (9), we obtain

$$(s-1)\exp(-H[\bar{\psi}_l]) \int D\hat{\psi}_l \exp(-\hat{\psi}_l M_{ll'} \hat{\psi}_{l'}) \tag{20}$$

where we have kept only quadratic terms and where the factor $(s-1)$ comes from the $(s-1)$ solutions $\bar{\psi}_l^{(\alpha)}$. The stability matrix $M_{ll'}(x, x')$ is given by

$$M_{ll'}^{(\alpha)} = [-\nabla^2 - \mu^2 + w(\bar{\psi}_1 + \bar{\psi}_2)]\delta_{ll'} - w[e_l^1 e_{l'}^1 \bar{\psi}_1 + e_l^\alpha e_{l'}^\alpha \bar{\psi}_2]. \tag{21}$$

The vectors e_l^1 and e_l^α span a two-dimensional subspace. The set of eigenfunctions of M with components in the $(s-2)$-dimensional subspace orthogonal to the (e_l^1, e_l^α) subspace will have eigenvalues large compared with zero. In the (e_l^1, e_l^α) subspace itself, there will be two sets of eigenfunctions: one with eigenvalues near zero, including the translational eigenfunctions $\phi_{l,i} = \mathrm{d}\bar{\psi}_l/\mathrm{d}x_{0,i}$ with zero eigenvalue, and a second set with eigenvalues large compared with zero. To find the exponent θ', we need only consider eigenvalues of M near zero (that is, those that tend to zero as $h \to 0$). So we can in this way ignore all but the first set of eigenvalues in the (e_l^1, e_l^α) subspace.

The problem of determining θ' thus reduces identically to the analogous problem for the Ising model considered by Langer (1967) and Günther *et al* (1980), and we find

$$\operatorname*{Im}_{|h|\to 0} F(-|h|) \sim \mu^d \left(\frac{\mu^{6-d}}{w^2}\right)^{d/2} \left(\frac{\mu^4}{|h|w}\right)^\rho \exp\left[-D\frac{\mu^{6-d}}{w^2}\left(\frac{\mu^4}{|h|w}\right)^{d-1}\right] \tag{22}$$

where

$$\rho = \begin{cases} \frac{1}{2}d(d-3), & d = 2, 4, 6, \\ \frac{7}{3}, & d = 3, \\ \frac{719}{90}, & d = 5. \end{cases} \tag{23}$$

This expression is valid as long as fluctuations near the percolation critical point can be ignored. If they are important, renormalisation group recursion relations near the critical point can be used to map the system to a renormalised length $\mu^{-1} \sim 1$. When this operation is carried out, we have (Houghton and Lubensky 1980)

$$\operatorname*{Im}_{|h|\to 0} F(-|h|) \sim \left(\frac{(p-p_c)^\Delta}{|h|}\right)^\rho \exp\left[-A\left(\frac{(p-p_c)^\Delta}{|h|}\right)^{d-1}\right]. \tag{24}$$

Substituting these expressions for $\operatorname{Im} F(-|h|)$ into equation (8) and evaluating the integral by steepest descent for large n gives equation (2) with

$$B(p-p_c) = A[(p-p_c)^\Delta]^\zeta \tag{25}$$

and

$$\theta' = \begin{cases} (1/2d)(1+4d-d^2), & d = 2, 4, 6, \\ -\frac{1}{9}, & d = 3, \\ -\frac{449}{450}, & d = 5, \end{cases} \tag{26}$$

so that $\theta' = \frac{5}{4}$ for $d = 2$, $\frac{1}{8}$ for $d = 4$ and $-\frac{11}{12}$ for $d = 6$.

Two comments regarding equation (26), the principal result of this paper, should be made. Firstly, θ' depends only on the dimension d, and is unrelated to any of the critical exponents associated with percolation. Secondly, the curious variation of the exponent θ' with dimension is due to the appearance of additional logarithmic divergences in the calculation of fluctuations about the critical droplet, which when exponentiated change the value of ρ in equation (22) (Günther *et al* 1980). These additional divergences exist only when d is an odd integer, hence the nature of the results for ρ and θ'. We have no physical interpretation for this behaviour at present.

The most important corrections to the form for $C(n)$ given by equation (2) come from the $O(h)$ corrections to the $|h|^{-d+1}$ power law in the exponential in equation (22). In terms of $\ln C(n)$ these give contributions proportional to $n^{1-2/d}$, $n^{1-3/d}$, ... in addition to the $n^{1-1/d}$ and $\theta' \ln n$ terms we have already discussed. The coefficients of these terms are non-universal and for $d > 2$ will be large compared with the prefactor. This may make it difficult to extract θ' from Monte Carlo data for any d other than two.

Finally, we note that the result (22) is to a large extent independent of the form of the interaction in (10). Thus the inclusion of quartic or higher-order terms will not change the exponent θ', and so the result can be expected to hold well away from p_c. More generally, the form (22) and the exponent ρ in equation (23) do not depend on the symmetry of the interaction (10) (except that it is discrete) only in the properties of the critical droplet. This universality has been discussed by Günther *et al* (1980), who also give arguments which suggest that the inclusion of two-loop and higher fluctuations about the critical droplet will leave our results unchanged.

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References

Fortuin C M and Kasteleyn P W 1972 *Physica* **57** 536
Günther N J, Nicole D A and Wallace D J 1980 *J. Phys. A: Math. Gen.* **13** 1755
Harris A B and Lubensky T C 1980 *Phys. Rev.* B in press
Houghton A and Lubensky T C 1980 *Phys. Lett.* **77A** 479
Kunz H and Souillard B 1978a *Phys. Rev. Lett.* **40** 133
—— 1978b *J. Stat. Phys.* **19** 77
Langer J S 1967 *Ann. Phys.* **41** 108
Lubensky T C 1979 in *Ill Condensed Matter* (*Les Houches 1978*) ed. R Balian, R Maynard and G Toulouse (New York: North-Holland)
Stauffer D 1979 *Phys. Rep.* **54** 1
Zia R K P and Wallace D J 1975 *J. Phys. A: Math. Gen.* **8** 1495